Spatial Voting with Endogenous Timing

by

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We consider a model of (spatial) voting with endogenous timing. In line with actual political campaigns, candidates can decide endogenously when and where to locate. More specifically, we analyze endogenous timing in a two-period n-candidate spatial-voting game. We show that this game possesses a pure-strategy equilibrium* (Osborne [1993]) but no – or only very complex – subgame-perfect equilibria. We demonstrate the latter point by analyzing the subgame-perfect equilibria in a *three*-candidate game. Our results show that allowing for endogenous timing can eliminate some of the more unappealing equilibrium characteristics of the standard model. (JEL: C 72, D 72, R 10)

1 *Introduction*

Among the many simplifying assumptions of the Downsian voting model, the (perhaps) most serious is that candidates select their positions simultaneously. In fact, assuming any specific exogenous order in which candidates select their platforms appears in conflict with the evidence. Typically, political campaigns last for several weeks, if not months, and candidates decide endogenously on when to take which stance in important matters. With the exception of Osborne's [1993], [2000] pioneering work, which analyzes endogenous timing with three candidates, the literature has remained remarkably quiet on this issue. In particular, we are not aware of general results for more than three candidates.[1]

This paper attempts to move a step forward in modeling endogenous timing in voting with an arbitrary number of candidates. Remarkably, introducing endogenous timing eliminates some of the less attractive properties of the standard model. The standard model predicts, for example, that with $n \geq 4$ parties, both outermost positions, the one farthest to the left and the one farthest to the right, are taken by two candidates *each*. Given that we are not aware of any countries with two parties on the far left and two parties on the far right, this appears to be a particularly unappealing

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[1] One notable exception is Osborne [1993], who also provides a result for $n = 4$ and $n = 5$ (after strengthening some of his assumptions).

feature of the standard model. With endogenous timing we can reconcile theory and empirics in that respect, since we find equilibria in which the positions farthest to the left and right are *not* taken by two candidates. Rather, candidates are more equally distributed over the political spectrum.

We model endogenous timing in very much the same way as it has been employed in the industrial organization literature (see, e.g., SALONER [1987], HAMILTON AND SLUTSKY [1990], or ROBSON [1990]). The general idea is the following. Suppose G is a normal-form game with n players whose strategy sets are $S_1, S_2, \dots, S_n$. Let G' be a two-period game in which all players act simultaneously in the first period. In the first period each player i chooses an action from S_i or chooses to wait. In period 2, all period 1 actions become common knowledge, and then all players who chose to wait in period 1 choose their actions simultaneously. Payoffs are assigned using the payoff matrix of the game G. Then G' is a game with endogenous timing. In this paper we investigate the equilibrium structure of a spatial-voting election game with endogenous timing. More particularly, we start with a standard spatial-voting election game in which n candidates simultaneously choose positions on the political spectrum [0, 1], along which voters are uniformly distributed. Each voter supports the nearest candidate. Each candidate's payoff is the length of the set of voters supporting her. Then we introduce endogenous timing as above.

It is natural to first consider subgame-perfect equilibria. However, it turns out that this is extremely complex. In the appendix we illustrate the difficulties of finding a subgame-perfect equilibrium (SPE) by only analyzing the special case with $n = 3$. We show that in this case the game has no SPE in pure strategies. However, there is a rather complex SPE in mixed strategies. On the equilibrium path of this mixed SPE, two candidates locate in period 1 at $1/4$ and $3/4$, respectively, whereas the third candidate locates somewhere in between these two positions in period 2. Due to the difficulties we encounter in finding a SPE in this special case, we consider an alternative equilibrium concept – the equilibrium* notion introduced by OSBORNE [1993]. Unlike a SPE, an equilibrium* is not required to be a Nash equilibrium on, or even to be defined on, every subgame. In particular, an equilibrium* is required to be a Nash equilibrium on any subgame that is reached by a history in the course of which no two players deviate in the same period from the play prescribed by the equilibrium*. It can be argued that an equilibrium* possesses the same stability that is the defining characteristic of subgame perfection.

We show that in the general n-player case, our voting model with endogenous timing has an equilibrium* in which any $n-1$ candidates locate at equidistant locations in period 1, whereas the nth candidate waits until period 2 and then locates somewhere in between the outermost positions occupied in the first period. While employing this procedure remedies the unintuitive features of the standard model, it also demonstrates, on a more technical level, the advantage and usefulness of the equilibrium* concept: An easy-to-prove equilibrium* may exist for a game that has no SPE, and there may be a simple equilibrium* for a game that has only complex or hard-to-find SPE.

Related Literature. The model we study is related to the ones in PALFREY [1984], OSBORNE [1993], and OSBORNE [2000]. Palfrey studies a three-candidate two-period model with exogenous timing in which two candidates locate in the first period and the third locates in the second period after observing the other two candidates' choices. Candidates are assumed to be vote maximizers. Palfrey shows that in this setup there exists a limit equilibrium in which the first-period candidates locate at different positions and the third candidate locates in between the two in the second period. Palfrey's and our model share the features of two location periods and vote-maximizing candidates. However, whereas Palfrey only considers the case of $n = 3$, we consider the n-player setup.[2] Moreover, whereas timing decisions are imposed exogenously in Palfrey's model, we endogenize this decision. Interestingly, restricting our result to the three-player case, Palfrey's timing and location pattern emerges endogenously in our model.

OSBORNE [1993] studies two versions of a three-candidate many-period model. If candidates move in a fixed order, as is assumed in one version, then there is a unique pure SPE outcome, in which the first candidate enters at the median voter's favorite position, the second candidate stays out, and the third candidate enters at the median. In another version of the model there is an infinite sequence of periods $1, 2, \ldots$ In each period every player who has not already chosen a position either chooses a position now or chooses to wait until the next period.[3] Given a strategy profile of the candidates, there is a date after which no new entry into the competition occurs. At this date an election is held. The winner is the candidate receiving most votes. Osborne shows that in every SPE of this model only one candidate enters and the other two stay out of the competition.

OSBORNE [2000] again uses the setup in which each of three candidates may move whenever she wishes. But this time candidates are uncertain about the distribution of the voters' favorite positions. Moreover, each candidate wishes to maximize the probability of winning, has a (symmetric) minimal acceptable probability of winning, p_0, and would rather stay out of the competition than enter and win with probability less than p_0. As Osborne shows, if there is sufficient uncertainty, "the game has an equilibrium (essentially a SPE) in which two players enter at distinct positions simultaneously in the first period and the third player either stays out, or, if there is enough uncertainty, enters in the second period at a position between those of the other candidates" (p. 42).

There are two main differences between Osborne's models and our approach. First, we allow for only two periods in which candidates may choose platforms, whereas Osborne allows for infinitely many. Second, and more importantly, we are not modeling a winner-take-all election, in which a payoff of, say, 1 for one candidate and 0 for all others would be appropriate, but a proportional-representation election, in which, for example, our candidates can be thought of as political parties, and

[2] Note that without proof Palfrey states the equilibrium configuration for the case of an arbitrary number of players locating in the first period and one more player locating in the second period.

[3] Note that by always choosing to wait, a candidate can stay out of the competition.

each party is awarded a number of seats in parliament proportional to the number of votes it receives.[4]

The paper is organized as follows. In section 2 we define spatial voting with endogenous timing. In section 3 we discuss the equilibrium* concept and show that the n-candidate spatial-voting model with endogenous timing has a pure-strategy equilibrium*. (As noted above, in the appendix we consider SPE in the three-player case.) Section 4 offers a short discussion of our findings and concludes with a short list of open problems.

2 *The Model*

For simplicity of exposition[5] we assume voters are uniformly distributed along [0, 1] with density 1. There are n (≥ 3) candidates, labeled $1, 2, ..., n$. The voting game consists of two periods. In period 1 all n candidates act simultaneously. For each i, candidate i chooses a position $x_i \in [0, 1]$ or chooses to wait until period 2. The choices then become common knowledge. In period 2 those candidates who chose to wait in period 1 simultaneously choose positions. After period 2, each occupied location x attracts all voters nearer to x than to any other occupied location.

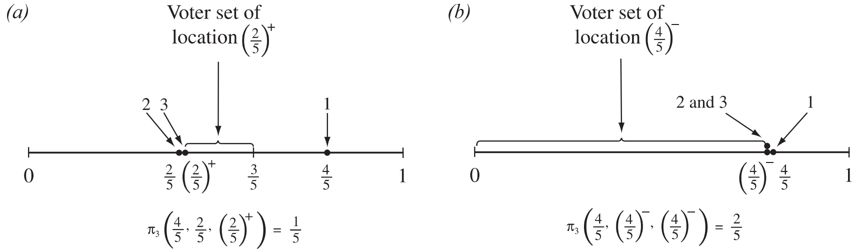
The set of voters attracted to x is called the voter set of location x. Then all candidates located at x share equally the voter set of location x. Thus, the payoff π_i to candidate $i = 1, 2, ..., n$ is the length of the voter set of location x_i divided by the number of candidates located at x_i. We assume that a candidate who locates at time $t = 2$ at an already occupied location x may specify x^+ or x^-. Without this standard assumption there is no hope for a Nash equilibrium even for many of the one-player subgames of the location game with endogenous timing. The examples in Figure 1 show how payoffs are assigned under the assumption.

Notice that the above notation is somewhat abbreviated: it can be inferred from $(4/5, 2/5, (2/5)^+)$ that candidate 2 located in period 1 and candidate 3 waited, then located in period 2, but the notation does not reveal whether candidate 1 located in period 1 or period 2. However, the notation is complete enough to determine payoffs. The assumption that a candidate locating in period 2 at an occupied location x may choose x^+ or x^- says that a candidate can adopt an opponent's political position, yet somehow signal that she is slightly to the left or to the right of that opponent. The assumption provides a nice compromise between the continuous voter distribution of our model and the fact that real elections have only finitely many voters.

[4] In an Hotelling-oligopoly model our assumption of vote-maximizing candidates naturally and realistically translates into demand-maximizing (and thus payoff-maximizing) firms when prices are fixed.

[5] It is easy to see that Propositions 2 and 4 (Appendix) can be proved in very much the same way for the case of any continuous, atomless distribution of voters.

Figure 1
Examples for Assignment of Payoffs



3 Analysis

The most natural way to analyze this game is to look for a SPE. However, it seems elusive to find a SPE for the general case of n players. This is illustrated in the appendix, where we consider the special case of $n = 3$. As it turns out, in this special case there is no *pure* SPE, and finding a SPE in *mixed* strategies is already extremely cumbersome. Therefore we consider the notion of an equilibrium* as defined in OSBORNE [1993]. Using this concept, we will prove much more easily that there are equilibria* in pure strategies in the general n-player case.

3.1 Equilibrium*

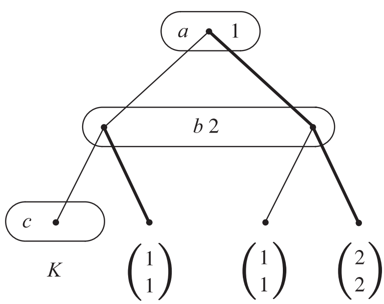
To circumvent the difficulties of fully specifying strategies for each of the players and of showing that these strategies constitute a (mixed) Nash equilibrium in each subgame, we proceed as in OSBORNE [1993], [2000] by applying the notion of an *equilibrium**. This notion requires only a partial specification of the players' strategies. The general idea of an equilibrium* is "that no player should be able to increase her payoff by changing her action in any period, given that the behavior of the players in the subgame to which the deviation leads is optimal" (OSBORNE [1993, p. 142]). As Osborne argues, "[t]o check that a strategy profile σ meets this condition, no information is needed about the behavior that σ prescribes in subgames that are reached when more than one player deviates from σ in some period."

More precisely, an equilibrium* is defined as follows: A substrategy σ_i of player i for a game G is a subset $H(\sigma_i)$ of the set of histories of G together with a function that assigns an action of player i to every history in $H(\sigma_i)$. More accurately, the function must assign an action of player i to $h \in H(\sigma_i)$ unless player i is not required to act in the period following h (in other words, unless player i has no information set in the game tree in the period following h). A profile σ of substrategies is an *equilibrium** if (1) for every player i, $H(\sigma_i)$ includes all histories that result when at most one player deviates from σ in any given period and (2) after any such history, no player can increase her payoff by a unilateral change of strategy, given that the other players continue to adhere to σ.

Clearly, including condition (1) in the definition of an *equilibrium** makes sure that the substrategies contain enough information to determine whether condition (2) is satisfied. Note that it is much easier to work with this notion of equilibrium than with SPE, since one does not have to worry about the existence of an equilibrium in subgames that can only be reached by a deviation of two or more players in a given period.

We now use two examples to illustrate the definition of equilibrium*. In Figure 2 subgame K begins at node c. Suppose K is a simultaneous-move game with no mixed-strategy Nash equilibrium. (For example, suppose players 1 and 2 simultaneously choose a positive integer, and the player who chooses the larger integer wins 10 while the other player wins 4. In case of a tie each wins 7.) Then G, the entire game, possesses no mixed-strategy SPE and consequently no pure-strategy SPE; however, s^* defined by the bold lines in Figure 2 is a pure-strategy equilibrium*. Notice that s^* satisfies condition (1) of the definition of equilibrium*, since the history of G leading to K requires that *two* players deviate from s^* in the first period of the game.

Figure 2
An Example



On the other hand, consider the game of Figure 2 altered slightly by making the information set labeled b into two information sets. Then s^* is not an equilibrium*, since K is the result of a history in which player 1 deviates in period 1 and player 2 then deviates in period 2 (note that by turning b into two information sets, we no longer have both players playing in period 1, but we now have player 1 playing in period 1 and player 2 playing in period 2). In fact the new game possesses no equilibrium*.

Concerning the relation between an equilibrium* and a SPE, note that a SPE is an equilibrium*, and that if every proper subgame has a SPE then any equilibrium* can be used as a starting point in the construction of a SPE. As was stated above, the strengths of the equilibrium* concept are first that an equilibrium* strategy profile possesses the same stability that is the defining characteristic of a SPE, and second that a simple equilibrium* may exist for a game with no SPEs or for a game that possesses only a complicated or hard-to-find SPE. We think this will become clear when comparing the results in the appendix with the ones that will be derived in the following subsection.

3.1.1 A Pure Equilibrium* for the n-Player Game

Consider the following substrategy profile s^*:[6]

(i) Candidates $1, 2, ..., n-1$ locate at $k, 3k, 5k, ..., 1-k$ with $k = 1/[2(n-1)]$, respectively, in the first period. After a history in which only candidate $i \in \{1, 2, \ldots, n-1\}$ deviates by choosing to wait until period 2, candidate i locates at the point he deviated from, $(2i-1)k$.

(ii) Candidate n chooses to wait in the first period and
(1) locates at $3k$ after a history in which the other candidates chose to locate as described in the first sentence of (i);
(2) locates at x^+ after a history in which the set of occupied positions is $\{x, 3k, 5k, ..., 1-k\}$ with $x < k$;
(3) locates at x^- after a history in which the set of occupied positions is $\{x, 3k, 5k, ..., 1-k\}$ with $k < x \leq 3k$;
(4) locates at $3k^-$ after a history in which the set of occupied positions is $\{x, 3k, 5k, ..., 1-k\}$ with $x > 3k$;
(5) similarly for a deviation in which locations $\{k, 3k, 5k, ..., 1-3k\}$ were chosen in the first period but location $1-k$ was not;
(6) locates at $(2i-1)k$ after a history in which the set of occupied positions is $\{x, k, ..., (2i-3)k, (2i+1)k, ..., 1-k\}$ with $x \neq (2i-1)k$, $i \in \{2, 3, ..., n-2\}$;
(7) locates at $(2i-1)k$ after a history in which only candidate $i \in \{1, 2, \ldots, n-1\}$ deviated by choosing to wait.

Proposition 1 will show that s^* is an equilibrium* of the voting game. The importance of this result is that it describes behavior that is much more in line with what we can observe in the field than the equilibria of the standard model. The occupied position farthest to the left, k, and the occupied position farthest to the right, $1-k$, are only occupied by one candidate each. The standard model predicts that these positions are occupied by two parties each. The difference is particularly striking for $n = 4$, a case not uncommon in elections worldwide.[7] The standard model with simultaneous moves predicts two left-wing parties at $1/4$ and two right-wing parties at $3/4$ while the center of the political spectrum is unoccupied. In contrast, s^* predicts one left-wing candidate who locates himself early at $1/6$, one right-wing candidate who chooses at $5/6$, a centrist candidate at $1/2$, and a fourth candidate who waits

[6] Of course, there is no loss of generality in letting candidates $1, 2, ..., n-1$ be the ones who commit in period 1 according to s^* and letting candidate n be the one who waits until period 2.

[7] For example, there are exactly four parties in the parliaments of Australia, Austria, Canada, and Hungary.

initially. In the case of s^* the fourth candidate also chooses a centrist position; but, as the proof of the next proposition will reveal, it is easy to construct other equilibria where he chooses either a right-wing or a left-wing platform. Thus, s^* predicts a wider variety of political platforms than one would expect in the standard model. And, perhaps most importantly, s^* predicts that there are centrist parties also with $n = 4$.

PROPOSITION 1 *The substrategy profile s^* is an equilibrium* of the voting game with endogenous timing.*

PROOF We will show that s^* is a Nash equilibrium and that s^* restricted to any subgame reached by a unilateral deviation is a Nash equilibrium of that subgame. We need consider only subgames reached by a single unilateral deviation, since there are only two periods in the game, so that a second unilateral deviation leads to a terminal node. Suppose G' is a subgame reached by a unilateral deviation. The deviation was not by candidate n, since a unilateral deviation by n leads to a terminal node, not a subgame. If G' is a result of $i \in \{1, 2, \dots, n-1\}$ waiting, then G' is a two-player subgame, both i and n locate at $(2i-1)k$, and each earns payoff k. Neither can do better by deviating unilaterally in G', that is, by relocating to $x \neq (2i-1)k$, since relocating to $x \in [k, 1-k]$ earns k, relocating to $x < k$ earns $(x+k)/2 < k$, and relocating to $x > 1-k$ earns $[(1-x)+k]/2 < k$. If G' is a result of candidate $i \in \{1, 2, \dots, n-1\}$ choosing in period 1 a location $x \neq (2i-1)k$, then G' is a one-player subgame, and that player is candidate n. We must show that s^* prescribes optimal behavior for n in G'. If $i \in \{2, 3, \dots, n-2\}$ then n locates at $(2i-1)k$. Candidate n has located in an unoccupied interval that is both as long as any other unoccupied interval and at least twice as long as either of the two outermost unoccupied intervals. Therefore n has maximized her payoff. If $i = 1$ and i locates at $x < k$, then n locates at x^+ in an unoccupied interval of length greater than $2k$, and the same argument holds. If $i = 1$ and i locates at x with $k < x < 3k$, then n locates at x^-. If $i = 1$ and i locates at $x \geq 3k$, then n locates at $3k^-$. In each case n has located in an outermost unoccupied interval of length greater than k. Candidate n has located so as to capture an outermost unoccupied interval that is at least half as long as any other unoccupied interval and at least as long as the other outermost unoccupied interval. Therefore n has maximized her payoff. The case $i = n-1$ is exactly symmetric.

It remains to show that s^* is a Nash equilibrium. Candidate n cannot increase his payoff by deviating unilaterally, since $\pi_n(x, s^*_{-n}) = k$ if $k \leq x \leq 1-k$; $\pi_n(x, s^*_{-n}) = (x+k)/2$ if $x < k$; $\pi_n(x, s^*_{-n}) = (1-x+k)/2 < k$ if $x > 1-k$; and $\pi_n(s^*) = k$. Similarly, if $i \in \{1, 2, \dots, n-1\}$ deviates unilaterally by waiting and locating at $x \in [0, 1]$ in period 2, or $i \in \{2, 3, \dots, n-2\}$ deviates unilaterally by locating at $x \neq (2i-1)k$ in period 1, then n locates at $(2i-1)k$, so that $\pi_i(x, s^*_{-i}) \leq k$ for all x while $\pi_2(s^*) = k$ and $\pi_i(s^*) = 2k$ for $i \neq 2$. Finally, if 1 deviates by locating at $x \neq k$ in period 1, then $\pi_1(x, s^*_{-1}) = x < k$ if $x < k$; $\pi_1(x, s^*_{-1}) = (3k-x)/2 < k$ if $k < x < 3k$; $\pi_1(x, s^*_{-1}) = k/2$ if $x = 3k$; $\pi_1(x, s^*_{-1}) = k$ if $3k < x \leq 1-k$; $\pi_1(x, s^*_{-1}) = (1-x+k)/2 < k$ if

$x > 1 - k$; and $\pi_1(s^*) = 2k$. Symmetrically, if $n-1$ deviates unilaterally by locating at $x \neq 1-k$ in period 1, then $\pi_{n-1}(x, s^*_{-(n-1)}) \leq k$ for all x, while $\pi_{n-1}(s^*) = 2k$.

3.1.2 Comparing Other Equilibria*

As indicated above, there are other equilibria* where one candidates waits and positions himself at a different location in period 2. Moreover, it is easy to see that there are also equilibria* where all candidates move in the first period – choosing the equilibrium locations of the standard simultaneous-move game. The reader may thus ask what we have really gained by introducing endogenous timing. We would like to offer several points: The model as such is more plausible. Its equilibrium structure is richer, and there exist some equilibria* that appear much more in line with evidence than the equilibria of the standard model. Moreover, comparing the different equilibria*, we find that the ones where one candidate waits in the first period not only yield a more realistic distribution of platforms but are also more stable. In particular, in an equilibrium* where all candidates move simultaneously, all candidates have alternative best responses (like choosing a location somewhere between other candidates or waiting and then choosing a location). In contrast, in the equilibrium* identified above, only players 1 and n have alternative best replies, while all candidates would be strictly worse off by changing their action. For $n = 3$ there is a pure equilibrium* where one candidate waits, but no pure equilibrium* where all candidates move simultaneously, which lets the equilibrium* identified above appear more parsimonious. Finally, let us reinterpret our model as one of firm behavior in an oligopoly market with product differentiation. When $n > 3$, we see that the equilibrium* in which one firm waits to locate is more efficient than the equilibria in the one-period model, as firms are spread out more evenly.

4 *Conclusion*

We have investigated a two-period spatial-voting game with endogenous timing. Our main result is that in the general n-player case, this game has an equilibrium* in which any $n-1$ candidates evenly spread out in period 1 whereas the nth candidate waits until period 2 and then chooses a platform.

On a purely game-theoretic level we have demonstrated the usefulness of the equilibrium* concept for sequential games as put forward by OSBORNE [1993]. Unlike a subgame-perfect equilibrium, an equilibrium* only requires a strategy vector to be a Nash equilibrium on any subgame that is reached by a history in the course of which no two players deviate in the same period from the play prescribed by the equilibrium*. This makes it much easier to work with. In fact, according to our analysis in the appendix, there is little hope of finding a (mixed) subgame-perfect equilibrium in the general n-player case of our spatial-voting model, whereas we showed with comparative ease that the general model does possess an equilibrium* in pure strategies.

On a substantive level, we consider our analysis another step toward more realistic voting models that allow for political campaigning during which candidates decide when and where to locate on the political spectrum. As we have shown here, making the assumptions more realistic can also make the equilibrium predictions more appealing. In particular, we have seen that with endogenous timing there are equilibria with more political variety and fewer extremist parties than the standard model predicts. Obvious avenues for future work are to allow for more periods or the analysis of continuous-time models. Also, it might be interesting to relax the assumption of perfect commitment by allowing candidates to relocate (at some cost) or analyzing reputation formation in repeated voting games. Finally, it would be worthwhile to see how the outcome of the model changes when voters are not uniformly distributed on the line [0, 1].

Appendix

A.1 Subgame-Perfect Equilibrium

In this appendix we demonstrate that it is elusive to find a SPE for the general n-player voting game with endogenous timing as introduced in section 2. Note that a strategy in an extensive-form game is a complete plan of behavior in the sense that it assigns an action to a player at each of his information sets. Thus, in the above voting game a strategy of a candidate is a tuple $(x_i^1, f_i(x_{-i}^1))$ where x_i^1 either specifies a position for period 1 or indicates that the candidate waits, *i.e.*, $x_i^1 \in [0, 1] \cup \{w\}$ with w indicating the decision to wait. The function $f_i(x_{-i}^1)$ is a mapping $\times_{j\neq i}([0, 1] \cup \{w\}) \rightarrow [0, 1]$ specifying the candidate's position choice in period 2 in response to what the other candidates did in period 1 in case she has decided to wait. So, a strategy in the voting game is a rather complicated mathematical object. But to fully specify a player's strategy is necessary if one wants to apply the notion of a SPE in order to solve the game.

A.1.1 A Mixed-Strategy Subgame-Perfect Equilibrium for the Three-Player Game

In this subsection we will consider the case $n = 3$. We will begin by studying a game called *Two Entrants and an Incumbent*. Two Entrants and an Incumbent is a two-person game. Voters are uniformly distributed on [0, 1] with density 1, and an incumbent is located at $a \in [0,\ 1]$. Then two entering candidates (the two players) locate simultaneously on $[0, a^-] \cup [a^+, 1]$.

PROPOSITION 2 *Two Entrants and an Incumbent has a pure-strategy Nash equilibrium if and only if* $a \in [0,\ 1/4] \cup [3/4,\ 1] \cup \{1/2\}$.

PROOF Suppose $a \in [3/4,\ 1]$. Let both entrants locate at $a/3$. Suppose candidate 1 relocates to x. If $0 \leq x < a/3$, then $\pi_1(x, a/3) < a/3 = \pi_1(a/3, a/3)$. If $a/3 < x \leq a^-$, then $\pi_1(x, a/3) = a/3 = \pi_1(a/3, a/3)$. If $a^+ \leq x \leq 1$, then $\pi_1(x, a/3) \leq 1 - a \leq 1 - 3/4 = 1/4 \leq a/3 = \pi_1(a/3, a/3)$. If $x = a$, then $\pi_1(x, a/3) = 1/2 - a/3 \leq 1/4 \leq \pi_1(a/3, a/3)$. Therefore,

$(a/3, a/3)$ is a Nash equilibrium if $a \in [3/4, 1]$. Similarly, if $a \in [0, 1/4]$, the strategy combination $(1-(1-a)/3, 1-(1-a)/3)$ is a Nash equilibrium, and if $a = 1/2$, then (a^-, a^+) is a Nash equilibrium. Next, suppose $a \in (1/2, 3/4)$ and candidates 1 and 2 locate at x_1 and x_2 respectively.

Case 1. $x_1 \in [a^+, 1] \cup \{a\}$ and $x_2 \neq a^-$. Then $\pi_2(x_1, a^-) = a > \pi_2(x_1, x_2)$.

Case 2. $x_1 \in [a^+, 1] \cup \{a\}$ and $x_2 = a^-$. Then $\pi_1(1-a, x_2) = (a+1-a)/2 = 1/2 > 1-a \geq \pi_1(x_1, x_2)$.

Case 3. $x_2 \in [a^+, 1] \cup \{a\}$. Proceed as in cases 1 and 2.

Case 4. $x_1, x_2 \in [0, a^-]$ and $x_1 \neq x_2$. Without loss of generality, $x_1 < x_2$. Then $\pi_1((x_1+x_2)/2, x_2) > \pi_1(x_1, x_2)$.

Case 5. $0 \leq x_1 = x_2 \leq a/3$. Then $\pi_1(a^+, x_2) = 1-a > 1/4 > a/3 \geq \pi_1(x_1, x_2)$.

Case 6. $a/3 < x_1 = x_2 \leq a^-$. Then $0 < (x_1+a)/4 < x_1 = x_2$, so that $\pi_1((x_1+a)/4, x_2) > (x_1+a)/4 = \pi_1(x_1, x_2)$.

Therefore, there is no pure-strategy Nash equilibrium if $a \in (1/2, 3/4)$ or, symmetrically, if $a \in (1/4, 1/2)$.

PROPOSITION 3 *Two Entrants and an Incumbent has a symmetric, continuous mixed-strategy Nash equilibrium when $a \in (1/4, 3/4)$.*

PROOF Fix $a \in (1/4, 3/4)$. Let

$$F(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq b \\ 1 + c(a-3x)^{-1/3} & \text{for } b \leq x \leq a^-, \\ k(2+a-3x)^{-1/3} & \text{for } a^+ \leq x \leq B, \\ 1 & \text{for } B \leq x \leq 1, \end{cases} \tag{A1}$$

where $0 < c < (2a)^{1/3}$, $b = (a+c^3)/3$, $0 < k < (2-2a)^{1/3}$, and $B = (2+a-k^3)/3$. Note that F, b, and B depend on c and k. We begin showing that for the proper choice of c and k, (F, F) is a mixed-strategy Nash equilibrium by establishing some facts about F. Using the definitions of c, b, k, and B,

$$a/3 < b < a < B < 1-(1-a)/3. \tag{A2}$$

By (A2), F is well defined. By (A1), F is increasing on $[b, a^-] \cup [a^+, B]$ and continuous on $[0, a^-] \cup [a^+, 1]$. To ensure that $F(a^-) = F(a^+)$, set $1 + c(-2a)^{-1/3} = k(2-2a)^{-1/3}$. Solving,

$$k = (2-2a)^{1/3} + c(1-1/a)^{1/3}. \tag{A3}$$

As c varies between its bounds 0 and $(2a)^{1/3}$, k varies between its bounds $(2-2a)^{1/3}$ and 0. Next, for c fixed, $\pi_1(y, F)$ is a constant function of y on (b, a^-), since

$$\frac{d\pi_1(y, F)}{dy} = 0 \text{ for } y \in (b, a^-),$$

as will now be shown:

$$\begin{aligned}\frac{d\pi_1(y,F)}{dy} &= d\bigg(\int_b^y ((a-x)/2)F'(x)dx \\ &\quad + \int_y^{a^-} ((x+y)/2)F'(x)dx + (1-F(a^-))(a+y)/2\bigg)\Big/dy \\ &= ((a-y/2)F'(y) - (y/2)F'(y) \\ &\quad + d((y/2)(F(a^-)-F(y)))/dy + (1-F(a^-))/2 \\ &= F'(y)(a/2-3y/2) - F(y)/2 + 1/2.\end{aligned}$$

By (A1), $F(y) = 1 + c(a-3y)^{-1/3}$ for $y \in (b, a^-)$ so that

$$\frac{d\pi_1(y,F)}{dy} = c(a-3y)^{-4/3}(a/2-3y/2) - (1+c(a-3y)^{-1/3})/2 + 1/2 = 0.$$

This completes the proof that for c fixed, $\pi_1(y, F)$ is a constant function of y on (b, a^-). In fact, the definition (A1) was derived by solving the differential equation $d\pi_1(y, F)/dy = 0$. Since F is continuous on $[0, a^-]$, $\pi_1(y, F)$ is a continuous function of y for $y \in [b, a^-]$, so that $\pi_1(y, F)$ is a constant function of y for $y \in [b, a^-]$. Also, it is clear that for $y \in [0, b)$, $\pi_1(y, F) < \pi_1(b, F)$, since with probability 1 candidate 2 locates in $[b, a^-] \cup [a^+, B]$, so that candidate 1 increases his voter set by moving from y to b. In summary, if candidate 1 is limited to playing $y \in [0, a^-]$, candidate 1 maximizes his payoff against F by choosing any $y \in [b, a^-]$. To show that candidate 1 limited to playing $y \in [a^+, 1]$ maximizes his payoff against F by choosing any $y \in [a^+, B]$, it is necessary to show $d\pi_1(y, F)/dy = 0$ for $y \in (a^+, B)$:

$$\begin{aligned}\frac{d\pi_1(y,F)}{dy} &= d\bigg(F(a^-)(1-(a+y)/2) \\ &\quad + \int_{a^+}^y (1-(x+y)/2)F'(x)dx + \int_y^B ((x-a)/2)F'(x)dx\bigg)\Big/dy \\ &= -F(a^-)/2 + (1-y/2)F'(y) \\ &\quad - d((y/2)(F(y)-F(a^+)))/dy - ((y-a)/2)F'(y) \\ &= F'(y)(2+a-3y)/2 - F(y)/2.\end{aligned}$$

By (A1), $F(y) = k(2+a-3y)^{-1/3}$ for $y \in (a^+, B)$ so that

$$\frac{d\pi_1(y,F)}{dy} = k(2+a-3y)^{-4/3}(2+a-3y)/2 - k(2+a-3y)^{-1/3}/2 = 0.$$

For $y \in (B, 1]$ we have $\pi_1(y, F) < \pi_1(B, F)$, since with probability 1 candidate 1 increases his market share by moving from y to B. If candidate 1 is limited to $y \in [a^+, 1]$, then candidate 1 maximizes his payoff by choosing any $y \in [a^+, B]$. To show that (F, F) is a Nash equilibrium for some choice of c, it is sufficient to show that for the proper choice of c, candidate 1 maximizes his payoff against F by locating in $[b, a^-] \cup [a^+, B]$. By our work above it is enough to show that for the proper choice of c we have $\pi_1(a^-, F) = \pi_1(a^+, F)$.

It is now convenient to introduce c into our notation and write $F_c(x)$ in place of $F(x)$. Fix $x_0 \in ({}^a/_3, a^-]$. As $c \to 0$, $b \to {}^a/_3$. Therefore $\lim_{c\to 0} F_c(x_0) = \lim_{c\to 0}[1 + c(a - 3x_0)^{-1/3}] = 1$. In other words, for small c player 2 playing F_c almost certainly locates near ${}^a/_3$. Then for small c

$$\text{(A4)} \qquad \pi_1(a^-, F_c) \approx a/3 < 1 - a \approx \pi_1(a^+, F_c).$$

Next fix $x_0 \in [a^+, 1 - (1-a)/3)$. As $c \to (2a)^{1/3}$, we have $k \to 0$ and $B \to (2 + a)/3 = 1 - (1-a)/3$. Therefore $\lim_{c\to(2a)^{1/3}} F_c(x_0) = \lim_{c\to(2a)^{1/3}} k(2 + a - 3x_0) = 0$. For c near $(2a)^{1/3}$, player 2 playing F_c almost certainly locates near $1 - (1-a)/3$. For c near $(2a)^{1/3}$

$$\text{(A5)} \qquad \pi_1(a^-, F_c) \approx a > (1-a)/3 \approx \pi_1(a^+, F_c).$$

Since $\pi_1(a^-, F) - \pi_1(a^+, F_c)$ is a continuous function of c, by (A4) and (A5) and the intermediate-value theorem there must be a $\overline{c} \in (0, (2a)^{1/3})$ such that $\pi_1(a^-, F_{\overline{c}}) = \pi_1(a^+, F_{\overline{c}})$. For this $\overline{c}$, $(F_{\overline{c}}, F_{\overline{c}})$ is a Nash equilibrium. *Q.E.D.*

We can now establish the claim made at the beginning of this section.

PROPOSITION 4 *The three-player location game with endogenous timing played on* $[0, 1]$ *with consumers distributed uniformly has a mixed-strategy subgame-perfect equilibrium, but no pure-strategy subgame-perfect equilibrium.*

PROOF By Proposition 2, if two players wait in period 1 and the third player locates at $a \in (1/4, 1/2) \cup (1/2, 3/4)$ in period 1, the resulting subgame has no pure-strategy Nash equilibrium. Therefore the three-player location game with endogenous timing has no pure-strategy SPE. Since the three-player location game with endogenous timing possesses an equilibrium* (s^* of Proposition 1), to show that it possesses a mixed-strategy SPE it is enough to show that every proper subgame possesses a SPE. SHAKED [1982] showed that the subgame that results when all three players wait in period 1 has a mixed-strategy Nash equilibrium. Each subgame that results when exactly two players wait in period 1 has a pure- or mixed-strategy Nash equilibrium by Propositions 2 and 3. It is quite easy to see that every subgame that results when exactly one player waits in period 1 has a pure-strategy Nash equilibrium: If players 1 and 2 locate at x_1 and x_2 respectively in period 1, then x_1^-, x_1^+, x_2^-, or x_2^+ is a Nash equilibrium of the one-player period-2 game. Since all the subgames in this paragraph are simultaneous-play games, all these Nash equilibria are trivially subgame-perfect. *Q.E.D.*

REMARK 1 *Note that on the equilibrium path of the mixed-strategy subgame-perfect equilibrium in Proposition 4, two candidates locate at* $1/4$ *and* $3/4$*, respectively, in period 1, whereas the third candidate locates at* $1/4^+$ *in period 2.*

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